

# Notes

## Recall: Divergence Theorem

If  $R$  is "nice" and  $\partial R$  is also nice and  $F$  is a v.f. on  $\mathbb{R}^3$  w/ its partial derivatives, then

$$\iint_{\partial R} F \cdot ds = \iiint_R \operatorname{div}(F) dV$$

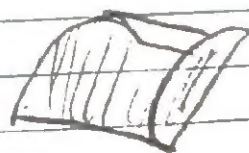
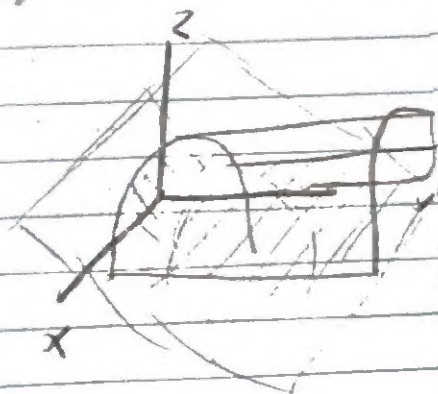
Ex: Compute the flux of  $F = \langle xy, y^2 + e^{xz^2}, \sin(xy) \rangle$  across  $S$  the surface of the region bounded by  $z = 1 - x^2$ ,  $z = 0$ ,  $y = 0$ ,  $y = z + 2$

Sol: Try to apply Div Thm

$$\begin{aligned} \operatorname{div}(F) &= \frac{\partial}{\partial x} [xy] + \frac{\partial}{\partial y} [y^2 + e^{xz^2}] \\ &\quad + \frac{\partial}{\partial z} [\sin(xy)] \end{aligned}$$

$$= y + 2y + 0 = 3y$$

$$R = \left\{ (x, y, z) : \begin{array}{l} -1 \leq x \leq 1 \\ 0 \leq z \leq 1 - x^2 \\ 0 \leq y \leq z + 2 \end{array} \right.$$



$$\begin{aligned} \iint_S F \cdot ds &= \iiint_{\partial R} F \cdot ds = \iiint_R \operatorname{div}(F) dV \\ &= 3 \int_{x=-1}^1 \int_{z=0}^{1-x^2} \int_{y=0}^{z+2} y \, dy \, dz \, dx \\ &= \frac{3}{2} \int_{-1}^1 \int_0^{1-x^2} [y^2]_{y=0}^{z+2} \, dz \, dx \end{aligned}$$

$$= \frac{3}{2} \int_{-1}^1 \int_0^{1-x^2} (2-z)^2 - 0 \, dz \, dx$$

$$= \frac{3}{2} \int_{-1}^1 \left[ -\frac{1}{3} (2-z)^3 \right]_0^{1-x^2} dx$$

$$= -\frac{1}{2} \int_{-1}^1 ((2-(1-x^2))^3 - (2-0)^3) dx$$

$$= -\frac{1}{2} \int_{-1}^1 ((1+x^2)^3 - 8) dx$$

$$= -\frac{1}{2} \int_{-1}^1 (3x^2 + 3x^4 + x^6 - 7) dx$$

$$= -\frac{1}{2} \left[ -7x + x^3 + \frac{3}{5}x^5 + \frac{1}{7}x^7 \right]_{-1}^1$$

$$= -\frac{1}{2} \left( -7 + 1 + \frac{3}{5} + \frac{1}{7} \right) \quad \square$$

Ex: Compute the flux of  $F = \langle xye^z, xyz^3, -ye^z \rangle$  across the box bounded by the four planes and  $x=3, y=2, z=1$

sol:  $R = [0, 3] \times [0, 2] \times [0, 1]$

Try to apply div thm

$$\text{div}(F) = ye^z + 2xyz^3 - ye^z = 2xyz^3$$

$$\therefore \iint_{\partial R} F \cdot d\mathbf{s} = \iiint_R \text{div}(F) \, dV = \rightarrow$$

$$2 \iiint_R xyz^3 dV$$

$$f(x) \cdot g(y) \cdot h(z)$$

$$= 2 \int_{x=0}^3 \int_{y=0}^2 \int_{z=0}^1 xyz^3 dz dy dx$$

constant

$$= 2 \left( \int_0^3 x dx \right) \left( \int_0^2 y dy \right) \left( \int_0^1 z^3 dz \right)$$

$$= 2 \left[ \frac{1}{2} x^2 \right]_0^3 \left[ \frac{1}{2} y^2 \right]_0^2 \left[ \frac{1}{4} z^4 \right]_0^1$$

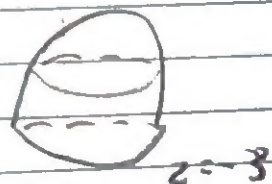
$$= \frac{1}{8} (3^2)(2^2)(1) = \frac{9}{2}$$

Ex: Compute Flux of  $F = \langle 2x^3 + y^3, y^3 + z^3, 3x^2z \rangle$  across surface of region bounded by paraboloid  $z = 1 - x^2 - y^2$  and plane  $z = -3$

Sol: Try to apply div thm

$$\begin{aligned} \text{div}(F) &= 6x^2 + 3y^2 + 3y^2 \\ &= 6(x^2 + y^2) \end{aligned}$$

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases}$$



shadow



$$R_{xy} = \{(r, \theta, z) : 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi, -3 \leq z \leq 1 - r^2\}$$

$$x^2 + y^2 = 4$$

$$\iint_R F \cdot ds = \iiint_R \operatorname{div}(F) dv$$

$\operatorname{div}(F)$

$$= \iiint_{R \cap \mathbb{R}^3} \operatorname{div}(F)_{\text{cyl}} r dz dr d\theta$$

$$= \int_0^{2\pi} \int_0^2 \int_{-3}^{1-r^2} 6r^2 dz dr d\theta$$

$$= 6 \int_0^{2\pi} \int_0^2 \left[ 2r^2 \right]_{-3}^{1-r^2} dr d\theta$$

$$= 6 \int_0^{2\pi} \int_0^2 r(r^2)(1-r^2 - (-3)) dr d\theta$$

$$= 6 \int_0^{2\pi} \int_0^2 (4r^3 - r^5) dr d\theta$$

$$= 6 \int_0^{2\pi} \left[ r^4 - \frac{1}{6} r^6 \right]_0^2 d\theta$$

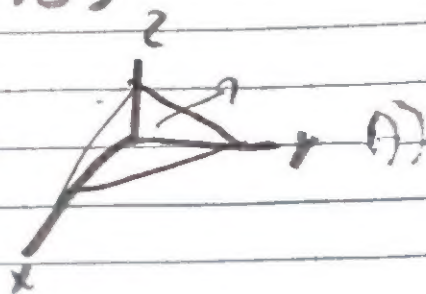
$$= 6 \int_0^{2\pi} \left( 16 - \frac{32}{3} - 0 \right) d\theta$$

$$= 32(2\pi - 0) = 64\pi$$

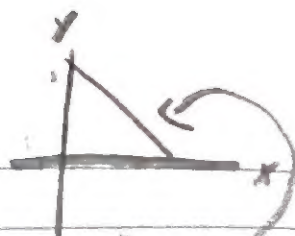
Ex: Compute the flux of  $F = \langle z, y, zx \rangle$  across surface of tetrahedron bounded by the coord planes and  $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$  (for unknown constants  $a, b, c > 0$ )

$$n \cdot (x-p) = 0$$

$$n \cdot x = n \cdot p = 0$$



$$n \cdot x = \alpha \quad n = \left\langle \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \right\rangle$$



$$R = \left\{ (x, y, z) : 0 \leq x \leq a, 0 \leq y \leq b\left(1 - \frac{x}{a}\right), \frac{1}{a} + \frac{y}{b} = 1, 0 \leq z \leq c\left(1 - \frac{x}{a} - \frac{y}{b}\right) \right\}$$

$\therefore$  Applying div thm

$$\text{div}(F) = 1+x$$

$$\iiint_R F \cdot ds = \iiint_R \text{div}(F) dV$$

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} (1+x) dz dy dx$$

$$= \int_0^a (1+x) \int_0^{b(1-\frac{x}{a})} \left[ z \right]_0^{1-\frac{x}{a}-\frac{y}{b}} dy dx$$

$$= \int_0^a (1+x) \int_0^{b(1-\frac{x}{a})} \left( c\left(1 - \frac{x}{a} - \frac{y}{b}\right) - 0 \right) dy dx$$

$$= \left( \int_0^a (1+x) \left[ y - \frac{xy}{a} - \frac{y^2}{2b} \right]_0^{b(1-\frac{x}{a})} dx \right)$$

$$= \left( \int_0^a (1+x) \left( b\left(1 - \frac{x}{a}\right) \left(1 - \frac{x}{a} - \frac{1}{2b}\left(1 - \frac{x}{a}\right)\right) - 0 \right) dx \right)$$

$$= bc \int_0^a (1+x) \left(1 - \frac{x}{a}\right) \left(1 - \frac{x}{a}\right) dx$$

$$= \frac{1}{2} bc \int_0^a \left( 1 + \left(1 - \frac{1}{a} - \frac{1}{a}\right)x + \left(-\frac{1}{a} - \frac{1}{a} + \frac{1}{a^2}\right)x^2 + \frac{1}{a^2}x^3 \right) dx$$

$$= \frac{1}{2} bc \int_0^a \left( 1 + \left( 1 - \frac{2}{a} \right) x + \left( \frac{1}{a^2} - \frac{2}{a} \right) x^2 + \frac{1}{a} x^3 \right) dx$$

$$= \frac{1}{2} bc \left[ x + \frac{1}{2} \left( 1 - \frac{2}{a} \right) x^2 + \frac{1}{3} \left( \frac{1}{a^2} - \frac{2}{a} \right) x^3 + \frac{1}{4a} x^4 \right]_0^a$$

$$= \frac{1}{2} bc \left( a + \frac{1}{2} \left( 1 - \frac{2}{a} \right) a^2 + \frac{1}{3} \left( \frac{1}{a^2} - \frac{2}{a} \right) a^3 + \frac{1}{4a} a^4 \right)$$

$$= \frac{1}{2} abc \left( 1 + \frac{1}{2} (a-2) + \frac{1}{3} (1-2a) \frac{1}{a} \right)$$

$$= \frac{1}{2} abc \left( \frac{1}{3} + a \left( \frac{1}{2} - \frac{2}{3} + \frac{1}{a} \right) \right) = \frac{1}{6} abc \left( 1 + \left( \frac{1}{4} \right) a \right)$$

$$= \frac{1}{6} abc \left( 1 - \frac{1}{4} a \right) \quad \square$$